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# The Blow-Up Time for Higher Order Semilinear Parabolic Equations with Small Leading Coefficients

AVNER FRIEDMAN AND LUC OSWALD*

*Department of Mathematics, Purdue University, West Lafayette, Indiana 47907*

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Consider the first initial-boundary value problem for parabolic equations such as

$$\mathscr{L}_\varepsilon u \equiv u_t + \varepsilon^4 \Delta^2 u = f(u),$$
$$\mathscr{L}_\varepsilon u \equiv u_t + \varepsilon^4 \Delta^2 u - \Delta u = f(u)$$

with $f(u)$ superlinear, and denote the blow-up time of the solution by $T_\varepsilon$. It is proved that $T_\varepsilon \to T_0$ as $\varepsilon \to 0$ where $T_0$ is the smallest blow-up time for $\mathscr{L}_0 u = f(u)$. Since there is no maximum principle for higher order parabolic equations, one cannot use here traditional comparison methods. The main results are outlined in Section 1. 

## 1. INTRODUCTION AND PRELIMINARY RESULTS

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with smooth boundary and let

$$Q_T = \Omega \times (0, T), \qquad \Omega_T = \Omega \times \{t = T\}.$$

Let $\phi(x)$ be a function satisfying

$$\phi \in C^1(\bar{\Omega}), \qquad \phi \geqslant 0, \qquad \phi \not\equiv 0, \qquad \phi = \partial\phi/\partial\nu = 0 \quad \text{on} \quad \partial\Omega, \tag{1.1}$$

where $\nu$ is the outward normal to $\partial\Omega$.

Let $f(u)$ be a nonlinear function defined on $\mathbb{R}^1$ and satisfying

$$\begin{gathered} f \in C^1(\mathbb{R}^1), \qquad f(u) \geqslant 0 \quad \text{if} \quad u \geqslant 0, \\ \liminf_{u \to \infty} \frac{f(u)}{u^p} \geqslant c \qquad \text{for some} \quad p > 1, \quad 0 < c \leqslant \infty. \end{gathered} \tag{1.2}$$

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For any $\varepsilon > 0$, consider the parabolic problem

$$\begin{aligned} &\frac{\partial u_\varepsilon}{\partial t} + \varepsilon^4 \Delta^2 u_\varepsilon = f(u_\varepsilon) \qquad \text{in} \quad Q_T, \\ &u_\varepsilon(x, 0) = \phi(x) \qquad \text{if} \quad x \in \Omega, \\ u_\varepsilon(x, t) = 0, \qquad &\frac{\partial}{\partial \nu} u_\varepsilon(x, t) = 0 \qquad \text{on} \quad \partial\Omega \times (0, T). \end{aligned} \tag{1.3}$$

DEFINITION 1.1. If a solution of (1.3) exists for $T < T_\varepsilon$ $(0 < T_\varepsilon < \infty)$ but not for $T < T_\varepsilon + \delta$, for any $\delta > 0$, then we call $T_\varepsilon$ the *blow-up time* for $u_\varepsilon$.

Notice that

$$\limsup_{T \to T_\varepsilon} \sup_{x \in \Omega} |u_\varepsilon(x, t)| = \infty. \tag{1.4}$$

Indeed, otherwise we can get a priori estimate on the $L^p(Q_{T_\varepsilon})$ norm of $D_t u_\varepsilon$, $D_x^4 u_\varepsilon$ for any $p < \infty$; consequently $u_\varepsilon$ is uniformly Hölder continuous in $Q_{T_\varepsilon}$ and then also $D_t u_\varepsilon$, $D_x^4 u_\varepsilon$ are uniformly Hölder continuous in $Q_{T_\varepsilon}$. But then the solution $u_\varepsilon$ can be extended into $Q_{T_\varepsilon + \delta}$ for some $\delta > 0$, which is a contradiction.

As a by-product of the results of Section 2 it will be shown that (1.4) can be improved, namely,

$$\lim_{T \to T_\varepsilon} \sup_{x \in \Omega} |u_\varepsilon(x, t)| = \infty. \tag{1.5}$$

In order to prove a finite-time blow-up for (1.3), for any $0 < \varepsilon \leqslant 1$, one can use, for instance, the method of proof of Theorem 4.1 or 4.2 and establish:

THEOREM 1.1. *If* $\liminf_{|u| \to \infty} (f(u)/|u|^p) > 0$ *for some* $p > 1$, *then for any compact subdomain* $K$ *of* $\Omega$ *there exists a positive number* $M$ *such that if* $\int_K \phi(x)\,dx > M$ *then* $T_\varepsilon < \infty$ *for any* $0 < \varepsilon \leqslant 1$.

This result will actually not be used in the rest of the paper, since our interest is in small values of $\varepsilon$.

Let $u$ be the solution of

$$\begin{aligned} &\frac{\partial u}{\partial t} = f(u), \qquad t > 0, \\ &u(x, 0) = \phi(x) \end{aligned} \tag{1.6}$$

and denote by $T(x)$ the blow-up time of $t \to u(x, t)$. Set

$$T_0 = T(x_0) \qquad \text{where} \quad \phi(x_0) = \max_{x \in \bar{\Omega}} \phi(x), \qquad x_0 \in \Omega. \tag{1.7}$$

We assume that

$$T_0 < \infty. \tag{1.8}$$

In Sections 2, 4 it will be proved that

$$T_\varepsilon \to T_0 \qquad \text{if} \quad \varepsilon \to 0. \tag{1.9}$$

In Section 2 we prove that $u_\varepsilon \to u$ uniformly in compact subsets of $\bar{\Omega} \times [0, T_0)$ and, consequently,

$$\liminf_{\varepsilon \to 0} T_\varepsilon \geqslant T_0. \tag{1.10}$$

In Section 3 we shall prove that higher order derivatives of $u_\varepsilon$ converge to the corresponding derivatives of $u$.

In Section 4 we shall establish the complement of (1.10), namely,

$$\limsup_{\varepsilon \to 0} T_\varepsilon \leqslant T_0; \tag{1.11}$$

in fact, we establish the stronger result:

$$T_\varepsilon \leqslant T_0 + C\varepsilon^{4/3}. \tag{1.12}$$

In Sections 5, 6 we consider the initial-boundary value problem for

$$\frac{\partial u_\varepsilon}{\partial t} + \varepsilon^4 \Delta^2 u_\varepsilon - \Delta u_\varepsilon = f(u_\varepsilon) \tag{1.13}$$

and compare the blow-up time $T_\varepsilon$ with the blow-up time $T_0$ for

$$\frac{\partial u}{\partial t} - \Delta u = f(u).$$

Using different methods than in Sections 2–4, we prove (1.10) (in Section 5) and (1.11) (in Section 6).

The conditions (1.1), (1.2) are assumed throughout the entire paper. In some of the sections we impose additional conditions on $f$, $\phi$.

The methods of this paper can be extended to semilinear parabolic equations of any order and to parabolic systems.

We finally mention that Friedman and Lacey [4] have established asymptotic estimates for $T_\varepsilon - T_0$ where $T_\varepsilon$ is the blow-up time corresponding to

$$\frac{\partial u_\varepsilon}{\partial t} - \varepsilon^2 \Delta u_\varepsilon = f(u_\varepsilon)$$

and $T_0$ is defined as in (1.7). They used comparison methods and constructed suitable subsolutions and supersolutions. Since there is no maximum principle for higher order equations, their methods cannot be extended to the present case. Friedman and Oswald [6] have recently derived asymptotic estimates on the blow-up surface $t = T_\varepsilon(x)$ for solutions of the hyperbolic equation

$$\frac{\partial^2 u_\varepsilon}{\partial t^2} - \varepsilon^2 \Delta u_\varepsilon = f(u_\varepsilon), \qquad n \leqslant 3.$$

## 2. Uniform Convergence of $u_\varepsilon$ to $u$

THEOREM 2.1. *Let $u_\varepsilon$ and $u$ be the solutions of* (1.3) *and* (1.6), *respectively, and let $0 < T < T_0$. Then, for all $\varepsilon$ small enough, $u_\varepsilon$ exists in $Q_T$ and*

$$u_\varepsilon \to u \text{ uniformly in } Q_T, \qquad \text{as} \quad \varepsilon \to 0.$$

COROLLARY 2.2. $\liminf_{\varepsilon \to 0} T_\varepsilon \geqslant T_0$.

In order to prove Theorem 1.1 we shall need the following:

LEMMA 2.3. *Let $0 < T < T_0$. If $u_\varepsilon$ exists in $Q_T$ and*

$$|u_\varepsilon|_{L^\infty(Q_T)} \leqslant C \qquad (0 < \varepsilon < 1)$$

*then*

$$u_\varepsilon \to u \text{ uniformly in } Q_T, \qquad \text{as} \quad \varepsilon \to 0; \tag{2.1}$$

*consequently, for all $\varepsilon$ sufficiently small,*

$$|u_\varepsilon|_{L^\infty(Q_T)} \leqslant M,$$

*where $M = 1 + |u|_{L^\infty(Q_T)}$.*

*Proof of Lemma* 2.3. It suffices to prove (2.1). We argue by contradiction: we suppose that (2.1) is not true so that, for some sequences $\varepsilon_i \downarrow 0$, $x_i \to x_0 \in \bar{\Omega}$, $t_i \to t_0 \in [0, T]$, there holds

$$|u_{\varepsilon_i}(x_i, t_i) - u(x_i, t_i)| \geqslant \delta > 0 \qquad \forall i. \tag{2.2}$$

Let $U_i(x, t) = u_{\varepsilon_i}(x_i + \varepsilon_i x, t)$. Then

$$\begin{gathered} \frac{\partial U_i}{\partial t} + \Delta^2 U_i = f(U_i), \\ U_i(x, 0) = \phi(x_i + \varepsilon_i x); \end{gathered} \tag{2.3}$$

$U_i$ is defined on $\Omega_i \times (0, t)$ where

$$\Omega_i = (\Omega - x_i)/\varepsilon_i,$$

and

$$|U_i|_{L^\infty(\Omega_i \times (0,t))} \leqslant C.$$

Further,

$$U_i(x, 0) \to \phi(x_0) \quad \text{uniformly in compact sets.}$$

Let us denote by $\Omega_\infty$ the limits set of the domains $\Omega_i$. Then

(i) if $\lim(\operatorname{dist}(x_i, \partial\Omega)/\varepsilon_i) = \infty$ then $\Omega_\infty = \mathbb{R}^n$,

(ii) if $\lim(\operatorname{dist}(x, \partial\Omega)/\varepsilon_i) < \infty$ then $\Omega_\infty$ is a half space and $x_0 \in \partial\Omega$.

By a priori estimates (up to a subsequence)

$$U_i \to U$$

uniformly in compact subsets of $\Omega_\infty \times [0, T]$ and, in the second case, $U$ satisfies zero boundary conditions.

In the first case

$$\begin{aligned} &U_t + \Delta^2 U = f(U) \qquad \text{in} \quad \mathbb{R}^n \times (0, T), \\ &U(x, 0) = \phi(x_0), \\ &U \text{ is bounded}, \end{aligned} \tag{2.4}$$

and in the second case

$$\begin{aligned} &U_t + \Delta^2 U = f(U) \qquad \text{in} \quad \Omega_\infty \times (0, T), \\ &U(x, 0) = \phi(x_0) = 0, \\ &U(x, t) = \frac{\partial}{\partial \nu} U(x, t) = 0 \qquad \text{on} \quad \partial\Omega_\infty \times (0, T), \\ &U \text{ is bounded}. \end{aligned} \tag{2.5}$$

Suppose we can prove that

$$\begin{aligned} &\text{the only solution of (2.4) is } U(x, t) = u(x_0, t), \\ &\text{and the only solution of (2.5) is } U(x, t) = 0 = u(x_0, t). \end{aligned} \tag{2.6}$$

Then, by (2.3) we deduce that

$$u_{\varepsilon_i}(x_i, t) = U_i(0, t) \to u(x_0, t)$$

uniformly in $t$, $0<t<T$, which contradicts (2.2). Thus it remains to prove (2.6).

Let $w(x, t)=U(x, t)-u(x_0, t)$. Then $w$ satisfies

$$w_t+\Delta^2 w=cw,$$
$$w(x, 0)=0,$$
$$w \text{ is bounded},$$

where $c$ is a bounded function and, in the second case,

$$w=\frac{\partial}{\partial \nu} w=0 \qquad \text{on} \quad \partial\Omega_\infty \times (0, T).$$

The function $w$ can be represented by the integral formula

$$w(x, t)=\int_0^t \int_{\Omega_\infty} K(x, t; \xi, s)\, c(\xi, s)\, w(\xi, s)\, d\xi\, ds, \tag{2.7}$$

where $K$ is the fundamental solution for the Cauchy problem in $\mathbb{R}^2$ in the first case, and Green's function for the Cauchy problem with Dirichlet data on the half space in the second case.

By [2, Chapter 4.2, (6)] the following estimate holds for Green's function $K$ corresponding to Dirichlet data on the boundary of any domain $\Omega_0 \subset \mathbb{R}^n$,

$$\int_{\Omega_0} |K(x, t; \xi, s)|\, d\xi \leqslant C \qquad \forall x \in \Omega_0, \quad 0<s<t<T; \tag{2.8}$$

here $\Omega_0$ may also be unbounded or $\mathbb{R}^n$.

Using (2.8) we deduce from (2.7) that

$$\begin{aligned} |w|_{L^\infty(\Omega_\infty \times (0, t))} &\leqslant C\, |w|_{L^\infty(\Omega_\infty \times (0, t))} \int_0^t \int_{\Omega_\infty} |K| \\ &\leqslant Ct\, |w|_{L^\infty(\Omega_\infty \times (0, t))}. \end{aligned}$$

Thus, for $t$ small enough, $w=0$ in $\Omega_\infty \times (0, t)$. Proceeding step by step we deduce that $w=0$ in $\Omega_\infty \times (0, T)$, and (2.6) follows.

*Proof of Theorem* 2.1. We first prove the existence of a solution $u_\varepsilon$ uniformly bounded in $Q_\sigma$ for $\sigma$ small enough. We introduce

$$v_\varepsilon(x, t)=u_\varepsilon\left(x, \frac{t}{\varepsilon^4}\right).$$

Then $v_\varepsilon$ satisfies

$$v_{\varepsilon,t} + \Delta^2 v_\varepsilon = \frac{1}{\varepsilon^4} f(v_\varepsilon).$$

For any $\sigma > 0$, set

$$A_\varepsilon = \{v \in L^\infty(Q_{\sigma\varepsilon^4});\, |v|_{L^\infty(Q_{\sigma\varepsilon^4})} \leqslant M\}, \qquad M > 0$$

and define the mapping $P_\varepsilon$:

$$P_\varepsilon \colon A_\varepsilon \to L^\infty(Q_{\sigma\varepsilon^4})$$
$$v_\varepsilon \to \bar{v}_\varepsilon$$

when $\bar{v}_\varepsilon$ is given by

$$\begin{aligned} &\bar{v}_{\varepsilon,t} + \Delta^2 \bar{v}_\varepsilon = \frac{1}{\varepsilon^4} f(v_\varepsilon) && \text{on} \quad Q_{\sigma\varepsilon^4}, \\ &\bar{v}_\varepsilon(x, 0) = \phi(x) && \text{in} \quad \Omega, \\ &\bar{v}_\varepsilon(x, t) = \frac{\partial}{\partial v} \bar{v}_\varepsilon(x, t) = 0 && \text{on} \quad \partial\Omega \times (0, \sigma\varepsilon^4). \end{aligned} \tag{2.9}$$

Representing $\bar{v}_\varepsilon$ by Green's function

$$\bar{v}_\varepsilon(x, t) = \int_\Omega K(x, t; \xi, 0)\, \phi(\xi)\, d\xi + \frac{1}{\varepsilon^4} \iint_{Q_t} K(x, t; \xi, s)\, f(v_\varepsilon(\xi, s))\, d\xi\, ds$$

and using (2.8), we get

$$|\bar{v}_\varepsilon|_{L^\infty(Q_{\sigma\varepsilon^4})} \leqslant C_0\, |\phi|_{L^\infty(\Omega)} + C_1 f_0(M) \sigma,$$

where $f_0(M) = \max\{|f(v)|, |v| \leqslant M\}$. If we choose

$$M = C_0\, |\phi|_{L^\infty(\Omega)} + 1 \tag{2.10}$$

then, for $\sigma \leqslant \sigma_0$, $P_\varepsilon$ maps $A_\varepsilon$ into itself, provided $\sigma_0$ is small enough. It is easy to check that for $\sigma$ small enough $P_\varepsilon$ is also a contraction. Therefore there exists one and only one fixed point of $P_\varepsilon$ in $A_\varepsilon$. In terms of $u_\varepsilon$ we have a unique solution on $Q_\sigma$ which is uniformly bounded. Notice that $\sigma$ depends only on $M$.

We now apply Lemma 2.3 to deduce that

$$u_\varepsilon \to u \text{ uniformly in } Q_\sigma, \qquad \text{as} \quad \varepsilon \to 0.$$

Hence if $\varepsilon$ is sufficiently small, say $\varepsilon \leqslant \varepsilon_1$,

$$|u_\varepsilon|_{L^\infty(Q_\sigma)} \leqslant |u|_{L^\infty(Q_\sigma)} + 1. \tag{2.11}$$

We next proceed to extend the solution $u_\varepsilon$ to $Q_{\sigma+\sigma_1}$, considering the initial values to be $u_\varepsilon(x, \sigma)$. In view of (2.11), if we choose

$$M = C_0(|u|_{L^\infty(Q_T)} + 1) + 1 \tag{2.12}$$

then as, in the first step, we can extend $u_\varepsilon$ by a fixed-point argument provided $\sigma_1$ is small enough. Applying Lemma 2.3 we deduce that (2.11) holds in $Q_{\sigma+\sigma_1}$ if $\varepsilon \leqslant \varepsilon_2$ and then, with the same choice of $M$ as in (2.12), we can extend the solution $u_\varepsilon$ to $Q_{\sigma+\sigma_1+\sigma_2}$, where we can take $\sigma_2 = \sigma_1$ since we have the same $M$ as in the second step.

Proceeding step by step, we extend the solution $u_\varepsilon$ into all of $Q_T$, and $u_\varepsilon \to u$ uniformly in $Q_T$.

*Remark* 2.1. One can easily show that (1.5) holds for any $\varepsilon > 0$. Indeed, otherwise

$$\sup_{x \in \Omega} |u_\varepsilon(x, t_j)| \leqslant C$$

for a sequence $t_j \uparrow T_\varepsilon$. By the first step in the proof of Theorem 2.1 we can then extend $u_\varepsilon$ into $Q_{t_j+\sigma}$ where $\sigma$ is a positive number independent of $t_j$. Since $t_j + \sigma > T_\varepsilon$ if $j$ is sufficiently large, this is a contradiction. The same remark applies to general nonlinear parabolic equations of the form

$$u_t - P(x, D)u = f(u).$$

## 3. Convergence of $\Delta^2 u_\varepsilon$ to $\Delta^2 u$

In this section we impose additional conditions on $f$, $\phi$:

$$f \in C^5(\mathbb{R}^2), \qquad f^{(j)}(0) = 0 \qquad \text{for} \quad 0 \leqslant j \leqslant 5, \tag{3.1}$$

$$\begin{gathered} \phi \in C^5(\bar{\Omega}), \\ \Delta^2\phi(x) = \frac{\partial}{\partial\nu}\Delta^2\phi(x) = 0 \qquad \text{for} \quad x \in \partial\Omega. \end{gathered} \tag{3.2}$$

THEOREM 3.1. *If* (3.1), (3.2) *hold then*

$$\Delta^2 u_\varepsilon \to \Delta^2 u \tag{3.3}$$

*uniformly on every compact subset of* $\bar{\Omega} \times [0, T_0)$.

The proof is somewhat analogous to the proof of Theorem 2.1. First we establish an analog of Lemma 3.2:

LEMMA 3.2. *Let $T < T_0$ and assume that*

$$|\Delta^2 u_\varepsilon|_{L^\infty(Q_T)} \leqslant C \qquad (0 < \varepsilon < 1). \tag{3.4}$$

*Then*

$$\Delta^2 u_\varepsilon \to \Delta^2 u \text{ uniformly in } \bar{Q}_T \tag{3.5}$$

*and, consequently,*

$$|\Delta^2 u_\varepsilon|_{L^\infty(Q_T)} \leqslant |\Delta^2 u|_{L^\infty(Q_T)} + 1 \tag{3.6}$$

*if $\varepsilon$ is sufficiently small.*

*Proof of Lemma* 3.2. The function $v_\varepsilon = \Delta^2 u_\varepsilon$ satisfies

$$v_{\varepsilon,t} + \varepsilon^2 \Delta^2 v_\varepsilon = f'(u_\varepsilon) v_\varepsilon + R(u_\varepsilon),$$

where $R(u_\varepsilon)$ is a sum of terms of the type

$$f^{(i)}(u_\varepsilon)\, D^{\alpha_1} u_\varepsilon \cdots D^{\alpha_j} u_\varepsilon$$

with $2 \leqslant i \leqslant 4$ and $\alpha_1 + \cdots + \alpha_j = i$. Let $W = \Delta^2 u$. If (3.5) is not true, then

$$|v_{\varepsilon_i}(x_i, t_i) - W(x_i, t_i)| \geqslant \delta > 0 \tag{3.7}$$

for sequences $\varepsilon_i \downarrow 0$, $x_i \to x_0 \in \bar{\Omega}$, $t_i \to t_0 \in [0, T]$. To derive a contradiction, let $V_\varepsilon(x, t) = v_\varepsilon(x_i + \varepsilon x, t)$. Then

$$V_{\varepsilon,t} + \Delta^2 V_\varepsilon = f'(u_\varepsilon) V_\varepsilon + R(u_\varepsilon),$$

where $f'(u_\varepsilon)$ and $R(u_\varepsilon)$ are evaluated at $(x_i + \varepsilon x, t)$, and

$$V_\varepsilon = \frac{\partial}{\partial v} V_\varepsilon = 0$$

on the lateral boundary. Moreover, $|V_\varepsilon| \leqslant C$.

We now proceed as in the proof of Lemma 2.3. Since, by (3.4), $f'(u_\varepsilon)$ and $R(u_\varepsilon)$ are uniformly Hölder continuous, by regularity results for parabolic equations (for instance, by the Schauder estimates) the sequence $V_{\varepsilon_i}$ has a subsequence such that

$$V_{\varepsilon_i}(x, t) \text{ uniformly convergent in compact subsets of } \Omega_\infty \times (0, T)$$
$$\text{to a function } V(x, t)$$

and

$$
\begin{aligned}
&V_t + \Delta^2 V = f'(u) V + R(u) && \text{in} \quad \Omega_\infty \times (0, T), \\
&V(x, 0) = \Delta^2 \phi(x_0) && \text{for} \quad x \in \Omega_\infty,
\end{aligned}
\tag{3.8}
$$

where $f'(u)$ and $R(u)$ are evaluated at $(x_0, t)$. The function $V$ is bounded in $\Omega_\infty \times (0, T)$ and either $\Omega_\infty = \mathbb{R}^n$ or $\Omega_\infty$ is a half space; in the second case $V$ and $\partial V/\partial \nu$ vanish on the lateral boundary.

Using the assumptions (3.1), (3.2) it is easy to check that the function $W(x_0, t)$ satisfies the same equations in (3.8) either in the whole space, or in the half space with the same homogeneous boundary conditions. We conclude as in Section 2 that $W \equiv V$, which is a contradiction to (3.7).

*Proof of Theorem* 3.1. We first establish that there exists a solution $u_\varepsilon$ is $Q_\sigma$ satisfying (3.4) with $T = \sigma$. For this purpose we introduce the function

$$
v_\varepsilon(x, t) = \Delta^2 u_\varepsilon\left(x, \frac{t}{\varepsilon^4}\right).
$$

Then

$$
\begin{aligned}
&v_{\varepsilon, t} + \Delta^2 v_\varepsilon = \frac{1}{\varepsilon^4} \Delta^2 f(v_\varepsilon) && \text{in} \quad Q_{\tau_\varepsilon}, \\
&v_\varepsilon(x, t) = \frac{\partial}{\partial \nu} v_\varepsilon(x, t) = 0 && \text{on} \quad \partial\Omega \times (0, \tau_\varepsilon), \\
&v_\varepsilon(x, 0) = \phi(x)
\end{aligned}
\tag{3.9}
$$

if $u_\varepsilon$ exists in $Q_{\tau_\varepsilon \varepsilon^4}$.

Analogously to Section 1 we introduce, for $\sigma > 0$, the set

$$
A_\varepsilon = \{w;\, w \in L^\infty(Q_{\sigma\varepsilon^4}),\, \Delta^2 w \in L^\infty(Q_{\sigma\varepsilon^4}),\, |\Delta^2 w|_{L^\infty(Q_{\sigma\varepsilon^4})} \leqslant M\}
$$

and the mapping

$$
\begin{aligned}
P_\varepsilon\colon A_\varepsilon &\to L^\infty(Q_{\sigma\varepsilon^4}) \\
w_\varepsilon &\to \bar{w}_\varepsilon,
\end{aligned}
$$

where $\bar{w}_\varepsilon$ satisfies the system (2.9) with $v_\varepsilon = w_\varepsilon$, $\bar{v}_\varepsilon = \bar{w}_\varepsilon$. Thus, applying $\Delta^2$ to the parabolic differential equation and representing $\Delta^2 \bar{w}_\varepsilon$ by Green's function, we obtain

$$
\begin{aligned}
\Delta^2 \bar{w}_\varepsilon(x, t) = &\int_\Omega K(x, t; \xi, 0)\, \Delta^2 \phi(\xi)\, d\xi \\
&+ \frac{1}{\varepsilon^4} \iint_{Q_t} K(x, t; \xi, s)\, \Delta^2 f(w_\varepsilon)\, d\xi\, ds,
\end{aligned}
$$

and $\Delta^2 f(w_\varepsilon) = f'(w_\varepsilon)\,\Delta^2 w_\varepsilon + R(w_\varepsilon)$ with $R(u)$ defined as before. We now easily see, using (2.8), that if

$$M = C_0\,|\Delta^2 \phi|_{L^\infty(\Omega)} + 1$$

and $\sigma$ is small enough then $P_\varepsilon$ maps $A_\varepsilon$ into itself and is a contraction. Thus, $P_\varepsilon$ has a unique fixed point in $A_\varepsilon$; this establishes the existence of a unique solution $u_\varepsilon$ in $Q_\sigma$ with

$$|\Delta^2 u_\varepsilon|_{L^\infty(Q_\sigma)} \leqslant M.$$

We now appeal to Lemma 3.2 in order to deduce that (3.5) holds in $Q_\sigma$ and

$$|\Delta^2 u_\varepsilon|_{L^\infty(Q_\sigma)} \leqslant |\Delta^2 u|_{L^\infty(Q_\sigma)} + 1 \tag{3.10}$$

if $\varepsilon$ is small enough. We can complete the proof of the theorem step by step, as in Section 2, always choosing the same $M$, namely

$$M = C_0(|\Delta^2 u|_{L^\infty(Q_T)} + 1) + 1$$

and the same size $t$-interval.

## 4. Convergence of $T_\varepsilon$ to $T_0$

In this section we complement Corollary 2.2 by proving:

THEOREM 4.1. *If* $\liminf_{|u| \to \infty} [f(u)/|u|^p] > 0$ $(p > 1)$, *then*

$$\limsup_{\varepsilon \to 0} T_\varepsilon \leqslant T_0. \tag{4.1}$$

Assuming that

$$\begin{aligned} &f(u) \text{ is convex} && \text{for} \quad u \in \mathbb{R}^1, \\ &f(u) \geqslant c\,|u|^p && \text{for all} \quad u \in \mathbb{R}^1 \text{ and some } c > 0,\ p > 1, \end{aligned} \tag{4.2}$$

we shall prove the stronger result:

THEOREM 4.2. *If* (4.2) *holds, then there exists a constant* $C > 0$ *such that*

$$T_\varepsilon \leqslant T_0 + C\varepsilon^{4/3}. \tag{4.3}$$

*Proof of Theorem* 4.2. Consider first the case where $f(u) = |u|^p$. We choose a function $\psi$ (cf. [1, p. 138]) such that

$$\psi = \chi^\alpha \qquad \text{where} \quad \alpha > 4, \quad \chi \in C_0^\infty(\mathbb{R}^n), \quad \chi \geqslant 0, \quad \int_{\mathbb{R}^n} \psi = 1 \tag{4.4}$$

and set

$$\psi_\rho(x) = \frac{1}{\rho^n} \psi\left(\frac{x - x_0}{\rho}\right), \qquad \rho > 0,$$

where $\rho$ is still small enough to be determined.

Expressing $\Delta^2 \psi$ in terms of $\chi$ and its derivatives, we find that

$$|\Delta^2 \psi| \leqslant C \psi^{1 - 4/\alpha}$$

and, consequently,

$$|\Delta^2 \psi_\rho| \leqslant C \psi_\rho^{1 - 4/\alpha} / \rho^{4(1 + n/\alpha)}. \tag{4.5}$$

Consider the function

$$g_\varepsilon(t) = \int u_\varepsilon(x, t)\, \psi_\rho(x)\, dx.$$

By (1.3),

$$g_\varepsilon'(t) = -\varepsilon^4 \int u_\varepsilon \Delta^2 \psi_\rho + \int |u_\varepsilon|^p \psi_\rho.$$

By (4.5)

$$-\int u_\varepsilon \Delta^2 \psi_\rho \leqslant C \int |u_\varepsilon| \frac{\psi_\rho^{1 - 4/\alpha}}{\rho^{4(1 + n/\alpha)}}.$$

We choose $\alpha$ large enough so that $(1 - 4/\alpha) > (1/p)$ and then, by Hölder's inequality,

$$\begin{aligned} -\int u_\varepsilon \Delta^2 \psi_\rho &\leqslant \mu \int |u_\varepsilon|^p \psi_\rho + \frac{C}{\mu^{1/(p-1)}} \int \frac{\psi_\rho^{(1 - 4/\alpha - 1/p)(p/(p-1))}}{\rho^{4(1 + n/\alpha)(p/(p-1))}} \\ &\leqslant \mu \int |u_\varepsilon|^p \psi_\rho + \frac{C}{\mu^{1/(p-1)} \rho^{4p/(p-1)}}. \end{aligned}$$

Hence

$$g_\varepsilon' \geqslant (1 - \mu\varepsilon^4) \int |u_\varepsilon|^p \psi_\rho - \frac{C\varepsilon^4}{\mu^{1/(p-1)} \rho^{4p/(p-1)}}, \tag{4.6}$$

or, by Jensen's inequality,

$$g_\varepsilon' \geqslant (1-\delta)\, g_\varepsilon^p - \gamma, \tag{4.7}$$

where

$$\begin{aligned} \delta &= \mu\varepsilon^4, \\ \gamma &= \frac{C\varepsilon^4}{\mu^{1/(p-1)}\rho^{p/(p-1)}}, \end{aligned} \tag{4.8}$$

and

$$g_\varepsilon(0) = \int u_\varepsilon(x, 0)\, \psi_\rho(x) \geqslant \phi(x_0) - C\rho^2$$

since $\phi(x) \geqslant \phi(x_0) - C\,|x - x_0|^2$ for some $C > 0$.

Setting

$$h(\tau) = Bg(t), \qquad \tau = At$$

with $1 - \delta = AB^{p-1}$ and $B(\phi(x_0) - C\rho^2) = \phi(x_0)$, we get

$$\begin{aligned} \frac{dh}{d\tau} &= h^p - \eta, \\ h(0) &= \phi(x_0) \end{aligned} \tag{4.9}$$

with

$$B \sim 1 + C\rho^2, \qquad A \sim 1 - \delta - C\rho^2 \tag{4.10}$$

and

$$\eta = \frac{B\gamma}{A} \sim \gamma(1 + \delta + C\rho^2). \tag{4.11}$$

The blow-up time $\tau_\varepsilon$ for (4.9) is given by

$$\tau_\varepsilon = \int_{\phi(x_0)}^{\infty} \frac{dh}{h^p - \eta} \tag{4.12}$$

and $T_\varepsilon \leqslant \tau_\varepsilon / A$. Also,

$$T_0 = \int_{\phi(x_0)}^{\infty} \frac{dh}{h^p} \tag{4.13}$$

and therefore

$$\tau_\varepsilon - \tau_0 \leqslant \int_{\phi(x_0)}^{\infty} \left( \frac{1}{h^p - \eta} - \frac{1}{h^p} \right) dh \leqslant C\eta.$$

It follows that

$$T_\varepsilon \leqslant T_0 < C(\gamma + \delta + \rho^2).$$

Choosing

$$\mu = C\rho^{4n/\alpha}, \qquad \rho^6 = \varepsilon^4$$

the assertion (4.3) follows.

The proof of the theorem for general $f$ satisfying (4.2) is similar. Indeed, from (4.6) we have

$$g_\varepsilon' \geqslant (1 - c\delta)\, f(g_\varepsilon)$$

provided $\gamma < c'\delta$ and $g_\varepsilon(0) = \phi(x_0) - C\rho^2$. Also

$$T_0 = \int_{\phi(x_0)}^{\infty} \frac{du}{f(u)},$$

$$(1 - 2\delta)\, T_\varepsilon \leqslant \int_{\phi(x_0) - C\rho^2}^{\infty} \frac{du}{f(u)},$$

so that

$$T_\varepsilon \leqslant T_0(1 - c\delta + C\rho^2).$$

Taking $\gamma = \delta = C\rho^2$ and $\rho^2 = \varepsilon$, (4.3) follows.

*Proof of Theorem* 4.1. Instead of (4.6) we now have

$$g_\varepsilon' \geqslant (c_1 - \varepsilon^4) \int |u_\varepsilon|^p\, \psi_\rho - C\left(1 + \frac{\varepsilon^4}{\rho^{4p/(p-1)}}\right)$$

and, by Jensen's inequality,

$$g_\varepsilon' \geqslant \frac{c_1}{2}\, g_\varepsilon^p - 2C \tag{4.14}$$

provided

$$\varepsilon < \rho^{p/(p-1)} \tag{4.15}$$

For any large constant $M > 0$ we can find $\delta > 0$ such that

$$u(x_0, T_0 - \delta) > 3M$$

and, by continuity,

$$u(x, T_0 - \delta) > 2M \qquad \text{if} \quad |x - x_0| < \rho_0.$$

In view of Theorem 2.1 we deduce that

$$u_\varepsilon(x, T_0 - \delta) > M \qquad \text{if} \quad |x - x_0| < \rho_0, \tag{4.16}$$

provided $\varepsilon$ is small enough. Now choose $\rho < \rho_0$ in the previous proof and $\varepsilon$ small enough so that (4.15) also holds. Thus (4.14) holds, and $g_\varepsilon(T_0 - \delta) \geqslant M$. It follows that $g_\varepsilon(t)$ blows up in time $\leqslant T_0 - \delta + \eta(M)$ where $\eta(M) \to 0$ if $M \to \infty$; this concludes the proof.

*Remark* 4.1. Using the method developed in Section 6, we shall extend (in Section 6) the assertion (4.1) to functions $f(u)$ which are not non-negative, such as

$$f(u) = |u|^{p-1} u.$$

*Remark* 4.2. The results of Sections 2–4 extend with minor changes to parabolic equations

$$u_t + \varepsilon^{2m} \Delta^m u = f(u).$$

## 5. Perturbation of Higher Order Equations

We wish to extend the results of the preceding sections to singular perturbation of nonlinear heat equations. We shall be primarily concerned with

$$\begin{aligned} &\frac{\partial u_\varepsilon}{\partial t} - \Delta u_\varepsilon + \varepsilon^4 \Delta^2 u_\varepsilon = f(u_\varepsilon) && \text{in} \quad Q_T, \\ &u_\varepsilon(x, 0) = \phi(x) && \text{if} \quad x \in \Omega, \\ &u_\varepsilon(x, t) = \frac{\partial}{\partial \nu} u_\varepsilon(x, t) = 0 && \text{on} \quad \partial\Omega \times (0, T). \end{aligned} \tag{5.1}$$

We denote by $T_\varepsilon$ the blow-up time of $u_\varepsilon$ (defined as in Definition 1.1); note that (1.5) holds (by Remark 2.1).

Let $u$ be the solution of

$$\begin{aligned} u_t - \Delta u &= f(u) && \text{in} \quad Q_{T_0}, \\ u(x, 0) &= \phi(x) && \text{if} \quad x \in \Omega, \\ u(x, t) &= 0 && \text{on} \quad \partial\Omega \times (0, T_0), \end{aligned} \tag{5.2}$$

where $T_0$ is the blow-up time for $u$; notice that $u > 0$ in $Q_{T_0}$. We assume that $T_0 < \infty$; if $f$ and $\phi$ are as in Theorem 1.1 then the proof of that theorem shows that $T_0 < \infty$.

We denote by $S_\varepsilon(t)$ the contraction semigroup of the operator $A_\varepsilon = \varepsilon^4 \Delta^2 - \Delta$ with domain $D(A_\varepsilon) = H^4(\Omega) \cap H_0^2(\Omega)$ (see [9]). Note that $D(A_\varepsilon)$ is independent of $\varepsilon$ and that the norm $\| \ \|_{H^{4m}}$ is equivalent to the graph norm of $D(A_\varepsilon^m)$.

We choose a positive integer $m$ such that

$$k = 4m > n + 4 \tag{5.3}$$

and assume that

$$\phi \in D(A_1^{m+1}) \tag{5.4}$$

and

$$f \in C^{k+4}, \qquad f^{(i)}(0) = 0 \qquad \text{for} \quad 0 \leqslant i \leqslant k + 4. \tag{5.5}$$

THEOREM 5.1. *Let $u$ be the solution of* (5.1), *and assume that* (5.3), (5.4) *hold. Then, for any $0 < T < T_0$, there exists a unique solution $u_\varepsilon$ of* (5.2) *in $Q_T$ with $u_\varepsilon(\cdot, t) \in C([0, t], H^k(\Omega))$, provided $\varepsilon$ is sufficiently small, and*

$$\sup_{0 \leqslant t \leqslant T} \|u_\varepsilon(\cdot, t) - u(\cdot, t)\|_{H^k(\Omega)} \to 0 \qquad \textit{if} \quad \varepsilon \to 0. \tag{5.6}$$

Recalling (5.3) and using the Sobolev imbedding theorem we conclude that $u_\varepsilon(x, t)$ is continuous in $\bar{Q}_T$.

COROLLARY 5.2. *If* (5.3), (5.4) *hold then*

$$\liminf_{\varepsilon \to 0} T_\varepsilon \geqslant T_0.$$

Theorem 5.1 is analogous to Theorems 2.1, 3.1; the proofs of these theorems do not extend to the present case of (5.1), although the underlying ideas are similar.

*Proof of Theorem* 5.1. For any $\sigma > 0$ we introduce the set

$$B = \{ w \in C([0, \sigma], D(A_1^m)); \|w\|_{H^k} \leqslant M \}$$

and consider the operator

$$P_\varepsilon\colon B \to C([0, \sigma]; H^k(\Omega))$$
$$w \to \bar{w},$$

where

$$\bar{w}(\cdot, t) = S_\varepsilon(t)\phi + \int_0^t S_\varepsilon(t-s) f(w(\cdot, s))\, ds. \tag{5.7}$$

It is easy to see that $S_\varepsilon(t)$ is a contraction on $D(A_1^m)$. Also, by (5.5), $f(w(\cdot, s))$ belongs to $D(A_1^m)$. Hence, applying $A_\varepsilon^m$ to both sides of (5.7) we deduce that

$$\|\bar{w}(\cdot, t)\|_{H^k} \leqslant C_0 \|\phi\|_{H^k} + C_1 \int_0^t \|f(w(\cdot, s))\|_{H^k}\, ds. \tag{5.8}$$

We claim that

$$\|f(w)\|_{H^k} \leqslant h(\|w\|_{H^k} + 1) \qquad \forall w \in D(A_1^m), \tag{5.9}$$

where $h(s)$ is a continuous increasing function of $s$. Indeed, the $k$th derivatives of $f(w)$ are sums of terms of the form

$$f^{(i)}(w) \prod_{j=1}^{n_i} D^{\gamma_j} w,$$

where $0 \leqslant i \leqslant k$, $\sum |\gamma_j| = k$. By Sobolev's imbedding,

$$|f^{(i)}(w)| \leqslant |f^{(i)}(\|w\|_{H^k})| + C \tag{5.10}$$

and

$$\|D^{\gamma_j} w\|_{L^{r_j}} \leqslant C \|w\|_{H^k} \qquad \text{where } \frac{1}{r_j} = \frac{1}{2} - \frac{k - \gamma_j}{n}.$$

Since

$$\sum \frac{1}{r_j} = n_i \left(\frac{1}{2} - \frac{k}{n}\right) + \frac{k}{n} \leqslant (n_i - 1)\left(\frac{1}{2} - \frac{k}{n}\right) + \frac{1}{2} \leqslant \frac{1}{2},$$

we get, Hölder's inequality,

$$\left| \prod_j D^{\gamma_j} w \right|_{L^2} \leqslant C(\|w\|_{H^k} + 1)^k.$$

Combining this with (5.10), the assertion (5.9) follows.

Substituting (5.9) into (5.8) and recalling the definition of the set $B$, we get

$$\|\bar{w}(\cdot, t)\|_{H^k} \leqslant C_0 \|\phi\|_{H^k} + C_1 \int_0^t h(M+1)\, dt.$$

If we choose

$$M = C_0 \|\phi\|_{H^k} + 1 \tag{5.11}$$

then, for $\sigma$ small enough, $P_\varepsilon$ maps $B$ into itself.

We claim that $f$ is Lipschitz continuous on $H^k$, that is,

$$\|f(w) - f(v)\|_{H^k} \leqslant C(M) \|w - v\|_{H^k} \qquad \text{if} \quad \|w\|_{H^k} \leqslant M, \quad \|v\|_{H^k} \leqslant M, \tag{5.12}$$

where $C(M)$ is a bounded function of $M$. Indeed, the $k$-derivatives of $f(w) - f(v)$ can be written as sum of terms of the form

$$(f^{(i)}(w) - f^{(i)}(v)) \prod_j D^{\alpha_j} w, \qquad \sum |\alpha_j| = k,$$

$$f^{(i)}(z) \left( \prod_j D^{\gamma_j} w \right) \left( \prod_j D^{\gamma'_j} v \right) D^\beta (w - v)$$

with $z = w$ or $z = v$ and $|\beta| + \sum |\gamma_j| + \sum |\gamma'_j| = k$. Estimating these terms as before (following (5.10)) we easily obtain the assertion (5.11).

Using (5.12) we can deduce from (5.7) that $P_\varepsilon$ is a contraction on $B$ provided $\sigma$ is sufficiently small, depending on $M$. Therefore $P_\varepsilon$ has a unique fixed point in $B$, which is then the unique solution $u_\varepsilon$ of (5.1) in $Q_\sigma$. We also have, as in (5.8),

$$\|u_\varepsilon(\cdot, t)\|_{H^{k+4}} \leqslant C_0 \|\phi\|_{H^{k+4}} + C \int_0^t \|f(u_\varepsilon(\cdot, s))\|_{H^{k+4}}\, ds \tag{5.13}$$

for $t < \sigma$.

We claim that

$$\|f(u_\varepsilon(\cdot, t))\|_{H^{k+4}} \leqslant h_0(\|u_\varepsilon(\cdot, t)\|_{H^k}) \|u_\varepsilon(\cdot, t)\|_{H^{k+4}} \tag{5.14}$$

when $h_0(s)$ is a continuous function of $s$. Indeed, the $(k+4)$th derivatives of $f(u_\varepsilon)$ are sums of terms of the form

$$f^{(i)} \prod_j D^{\alpha_j} u_\varepsilon,$$

where $1 \leqslant i \leqslant k+4$, $\sum |\alpha_j| = k+4$. Pick up $j_0$ with the maximum $|\alpha_{j_0}|$. Since

$\sum |\alpha_j| = k+4$, we then have $|\alpha_j| \leqslant k/2+2$ for all $j \neq j_0$ and, by Sobolev's imbedding and (5.3),

$$\|D^{\alpha_j} u_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leqslant C \|u_\varepsilon(\cdot, t)\|_{H^k(\Omega)} \qquad \text{if} \quad j \neq j_0.$$

From this we clearly deduce that (5.14) is valid.

Substituting (5.14) into (5.13) we find that for any $M > 0$,

$$\begin{aligned} &\text{if } \|u_\varepsilon(\cdot, t)\|_{H^k} \leqslant M \text{ for } t \leqslant \sigma \text{ then} \\ &\|u_\varepsilon(\cdot, t)\|_{H^{k+4}} \leqslant C_0 \|\phi\|_{H^{k+4}} + h_1(M) \int_0^t \|u_\varepsilon(\cdot, s)\|_{H^{k+4}}\, ds, \end{aligned} \tag{5.15}$$

where $h_1(M)$ is a continuous function of $M$; consequently, by Gronwall's inequality,

$$\begin{aligned} &\text{if } \|u_\varepsilon(\cdot, t)\|_{H^k} \leqslant M \text{ for } t \leqslant \sigma \text{ then} \\ &\|u_\varepsilon(\cdot, t)\|_{H^{k+4}} \leqslant h_2(M) \end{aligned} \tag{5.16}$$

with another continuous function $h_2(s)$; the function $h_2$ is independent of $\sigma$.

We can apply (5.16) with $M$ as in (5.11) and then, by compactness, deduce that

$$\sup_{0 \leqslant t \leqslant \sigma} \|u_\varepsilon(\cdot, t) - u(\cdot, t)\|_{H^k(\Omega)} \to 0 \qquad \text{if} \quad \varepsilon \to 0. \tag{5.17}$$

It follows that

$$\|u_\varepsilon(\cdot, \sigma)\|_{H^k} \leqslant \sup_{0 \leqslant t \leqslant T} \|u(\cdot, t)\|_{H^k} + 1$$

if $\varepsilon$ is sufficiently small.

We can now extend the solution $u_\varepsilon$ to $Q_{2\sigma}$, replacing

$$\|\phi\|_{H^k} \text{ by } \sup_{0 \leqslant t \leqslant T} \|u(\cdot, t)\|_{H^k} + 1$$

in the definition of $M$ in (5.11). Next we establish (5.16) for $t < 2\sigma$, and continue as before to derive (5.17) for $0 \leqslant t \leqslant 2\sigma$. Proceeding step by step (cf. also the proof in Section 2) we can reach $t = T$ after a finite number of steps; thus $u_\varepsilon$ exists in $Q_T$ and (5.6) holds.

## 6. Convergence of $T_\varepsilon$ to $T_0$

In this section we complement Corollary 5.2 by establishing that $\limsup T_\varepsilon \leqslant T_0$. We assume that

$$\lim_{|u|\to\infty} \frac{uf(u)}{|u|^{p+1}} = \gamma \qquad \text{where} \quad \gamma > 0, \quad p > 1 \tag{6.1}$$

and that either

$$p < \frac{n+4}{n} \tag{6.2}$$

or

$$\begin{aligned}
&\text{(a)} \quad \Omega \text{ is convex}, \ \Delta\phi + f(\phi) \geqslant 0,\\
&\text{(b)} \quad f^{\vartheta}(u) \text{ is convex for } u \geqslant 0 \text{ and some } 0 < \vartheta < 1,\\
&\text{(c)} \quad f \text{ and } \phi \text{ satisfy (5.4), (5.5)},\\
&\text{(d)} \quad |f(u) - \gamma u^p| = O(u^{p-\delta}) \text{ for } u \to +\infty \text{ and some } \delta > 0,\\
&\text{(e)} \quad p < \frac{n+2}{n-2}.
\end{aligned} \tag{6.3}$$

THEOREM 6.1. *If* (6.1) *and either* (6.2) *or* (6.3) *hold, then the blow-up times* $T_\varepsilon$, $T_0$ *of the solutions of* (5.1), (5.2) *satisfy*

$$\limsup_{\varepsilon\to 0} T_\varepsilon \leqslant T_0. \tag{6.4}$$

*Proof.* Multiplying (5.1) by $u_\varepsilon$ and integrating over $\Omega_t$, we get

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_t} u_\varepsilon^2\,dx + \int_{\Omega_t} (\varepsilon^4(\Delta u_\varepsilon)^2 + |\nabla u_\varepsilon|^2) = \int_{\Omega_t} u_\varepsilon f(u_\varepsilon). \tag{6.5}$$

Next, multiplying (5.1) by $u_{\varepsilon,t}$ and integrating over $\Omega_t$, we get

$$\int_{\Omega_t} u_{\varepsilon,t}^2 + \frac{1}{2}\frac{d}{dt}\int_{\Omega_t} (\varepsilon^4(\Delta u_\varepsilon)^2 + |\nabla u_\varepsilon|^2) = \frac{d}{dt}\int_{\Omega_t} F(u_\varepsilon), \tag{6.6}$$

where

$$F(r) = \int_0^r f(s)\,ds.$$

Integrating (6.6) over $(0, t)$ we obtain the inequality

$$\frac{1}{2}\int_{\Omega_t} (\varepsilon^4(\Delta u_\varepsilon)^2 + |\nabla u_\varepsilon|^2) \leqslant \int_{\Omega_t} F(u_\varepsilon) + C. \tag{6.7}$$

Using this in (6.5) results in

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_t} u_\varepsilon^2 \geqslant \int_{\Omega_t} [u_\varepsilon f(u_\varepsilon) - 2F(u_\varepsilon)] - C$$

and, by (6.1),

$$\frac{d}{dt}\int_{\Omega_t} u_\varepsilon^2 \geqslant c\int_{\Omega_t} |u_\varepsilon|^{p+1} - C \qquad (c > 0, C > 0). \tag{6.8}$$

We now proceed to establish (6.4) by contradiction. Suppose

$$T_\varepsilon \geqslant T_0 + \alpha \qquad \text{for some} \quad \alpha > 0$$

and a sequence $\varepsilon \downarrow 0$. Consider two cases:

*Case* 1. $\int_{\Omega_t} u_\varepsilon^2(x, t) \to \infty$ for some $t = \tau_\varepsilon \leqslant T_0 + \alpha/2$ and a subsequence $\varepsilon \to 0$. Since, by (6.8),

$$\frac{d}{dt}\int_{\Omega_t} u_\varepsilon^2 \geqslant c_1\left(\int_{\Omega_t} u_\varepsilon^2\right)^{(p+1)/2} - C \qquad \text{for} \quad t \geqslant \tau_\varepsilon,$$

where $c_1$ is a positive constant, we deduce that $\int_{\Omega_t} u_\varepsilon^2$ must blow-up at time $t = \tau_\varepsilon + \alpha/4$ if $\varepsilon$ is small enough, which is a contradiction since $\tau_\varepsilon + \alpha/4 < T_\varepsilon$.

*Case* 2 (This is the complement of Case 1). $\int_{\Omega_t} u_\varepsilon^2(x, t) \leqslant C$ for all $t \leqslant T_0 + \alpha/2$ and $\varepsilon$ small enough. Integrating (6.8) we get

$$\iint_{Q_{t_0+\alpha/2}} |u_\varepsilon|^{p+1} \leqslant C. \tag{6.9}$$

Next, integrating (6.5) once in $t$ we get

$$\iint_{Q_{T_0+\alpha/2}} |\nabla u_\varepsilon|^2 \leqslant C \tag{6.10}$$

and integrating (6.6) twice in $t$ we get, after using (6.9),

$$\int_0^{T_0+\alpha/2} \left(T_0 + \frac{\alpha}{2} - t\right) dt \iint_{Q_t} u_{\varepsilon,t}^2 \leqslant C,$$

so that

$$\iint_{Q_{T_0+\alpha/4}} u_{\varepsilon,t}^2 \leqslant C. \tag{6.11}$$

From (6.10), (6.11) we deduce that, for a subsequence,

$$\begin{aligned} &u_\varepsilon \to v \qquad \text{a.e. in } Q_{T_0+\alpha/4}, \\ &u_\varepsilon \to v \qquad \text{strongly in } L^2(Q_{T_0+\alpha/4}). \end{aligned} \tag{6.12}$$

Further, from (6.12), (6.9),

$$\iint_{Q_{T_0+\alpha/4}} |f(u_\varepsilon)|^{(p+1)/p} \leqslant C, \tag{6.13}$$

$$\iint_{Q_{T_0+\alpha/4}} |f(v)|^{(p+1)/p} \leqslant C. \tag{6.14}$$

It follows that the functions $|f(u_\varepsilon)-f(v)|$ are uniformly integrable and converge a.e. to zero, as $\varepsilon \to 0$. Consequently

$$\iint_{Q_{T_0+\alpha/4}} |f(u_\varepsilon)-f(v)| \to 0 \qquad \text{if} \quad \varepsilon \to 0. \tag{6.15}$$

From (6.12), (6.15) and (6.9), (6.10) it follows that $v$ is a strong solution of (5.2) in $Q_{\tau 0+\alpha/4}$.

By the $L^r$ estimates of the parabolic equations [8] it then follows that

$$\iint_{Q_{T_0+\alpha/4}} (|v_t|^r + |\nabla_x v|^r) \leqslant C \iint_{Q_{t_0+\alpha/4}} |f(v)|^r + C < \infty$$

for $r = (p+1)/p$ and then, by Sobolev's imbedding (see [8; p. 80])

$$\iint_{Q_{T_0+\alpha/4}} |v|^q < \infty \qquad \text{for} \quad \frac{1}{q} = \frac{1}{r} - \frac{2}{n+2}.$$

If we assume that (6.2) holds then $q > p+1$ and we can repeat the above argument a finite number of times to deduce that $v \in L^\infty(Q_{T_0+\alpha/4})$. Since, by uniqueness, $u \equiv v$ in $Q_{T_0}$, this is a contradiction.

Consider next the case where (6.3) holds. Then by Theorem 5.1 (here we need (6.3)(c)) $u_\varepsilon \to u$ uniformly in $Q_{T_0}$ and therefore $u \equiv v$ in $Q_{T_0}$. In view of (6.11) we then have

$$\iint_{Q_{T_0}} u_t^2 < \infty. \tag{6.16}$$

By [5] (see also [3]; the conditions (6.3)(a), (6.3)(b) are needed here) $u$ does not blow up near the boundary $\partial\Omega$, and

$$u_t \geqslant cf(u) \qquad \text{in} \quad K\times(0, T_0) \quad (c>0)$$

for any compact subdomain $K$ of $\Omega$. Take any blow-up point $x_0$ in $\Omega$ contained in the interior of such a set $K$. By a recent result of Giga and Kohn [7]

$$u(x, t) \geqslant c_0(T_0-t)^{-1/(p-1)} \qquad \text{if} \quad |x-x_0| \leqslant C\sqrt{T-t}$$

for any positive constant $C$ where $c_0$ is a positive constant depending on $C$ (here we need the assumption (6.3)(d) and $p \leqslant (n+2)/(n-2)$). It follows that, for some $c_1 > 0$,

$$\iint_{Q_{T_0}} u_t^2 \geqslant c_1 \int_0^{T_0} (T_0-t)^{-2p/(p-1)}\,(T_0-t)^{n/2}\,dt = \infty$$

since $p < (n+2)/(n-2)$; this is a contradiction to (6.16).

The proof of Theorem 6.1 can be adapted also to the case (1.3), (1.6) and it yields an extension of the results of Section 4 to the case where $f(u)$ satisfies

$$\lim_{|u|\to\infty} \frac{uf(u)}{|u|^{p+1}} = \gamma > 0, \qquad f(u) > 0 \quad \text{if } u > 0. \tag{6.17}$$

We shall also assume that

$$\phi(x) \geqslant \phi(x_0) - c\,|x-x_0|^m \text{ in some neighborhood of } x_0 \ (c>0, m>0) \tag{6.18}$$

and that

$$p < 1 + \frac{2m}{n}. \tag{6.19}$$

THEOREM 6.2. *Denote by $T_\varepsilon$ and $T_0$ the blow-up times for* (1.3) *and* (1.6), *respectively. If* (6.17)–(6.19) *hold then*

$$\limsup_{\varepsilon\to 0} T_\varepsilon \leqslant T_0.$$

*Proof.* As in the previous proof we derive the inequality (6.8). Since $u_\varepsilon \to u$ uniformly in $Q_{T_0-\delta}$ for any $\delta > 0$ (by Section 1), if we can prove that

$$\int_{\Omega_t} u^2 \to \infty \qquad \text{as} \quad t \to T_0 \tag{6.20}$$

then we shall have, for any large constant $M$,

$$\int_\Omega u_\varepsilon^2(x, T_0-\delta) > M$$

if $\delta$ is small enough, provided $\varepsilon$ is sufficiently small, say $\varepsilon \leqslant \varepsilon_0(\delta)$. Using (6.8) we can then deduce that $\int_{\Omega_t} u_\varepsilon^2$ blows up in time $\leqslant T_0-\delta+\eta(M)$ when $\eta(M)\to 0$ if $M\to\infty$; but this completes the proof of the theorem. Thus it remains to establish (6.20).

For simplicity, consider first the case $f(u)=|u|^{p-1}u$. Then

$$u(x,t)=\frac{\phi(x)}{[1-(p-1)\phi^{p-1}(x)t]^{1/(p-1)}} \tag{6.21}$$

and

$$T_0=\frac{1}{(p-1)\phi^{p-1}(x_0)}.$$

Set $t=T-\delta$, $r=|x-x_0|$. Then, near $x=x_0$,

$$u(x,t)\geqslant C(\delta+r^m)^{-1/(p-1)} \qquad \text{by (6.21)},$$

so that

$$\int_\Omega u^2(x,t)\,dx \geqslant \int_0^{R_0}\frac{Cr^{h-1}\,dr}{(\delta+r^m)^{2/(p-1)}}\to\infty$$

when $t\to T$, since

$$\frac{2m}{p-1}-n+1>1 \qquad \text{by (6.19)}.$$

This completes the proof of (6.20) in case $f(u)=|u|^{p-1}u$.

For general $f$, (6.21) is replaced by

$$\int_{\phi(x)}^{u(x,t)}\frac{dv}{f(v)}=t$$

and we can proceed in a similar way.

*Remark* 6.1. The results of Sections 5, 6 can be extended to parabolic equations of any order; for instance, to

$$u_t-\Delta u+\varepsilon^{2m}\Delta^m u=f(u),$$

where $m$ is any integer $\geqslant 3$.

*Remark* 6.2. The results of this paper can be extended to systems of parabolic equations, such as, for instance,

$$u_t - \gamma\,\Delta u + \varepsilon^4 \Delta^2 u = f(u, v),$$

$$v_t - \delta\,\Delta v + \varepsilon^4 \Delta^2 v = g(u, v),$$

where $\gamma$, $\delta$ are constants, $\gamma \geqslant 0$, $\delta \geqslant 0$.

## References

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